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Conformal symmetry of an extended Schrödinger equation and its relativistic origin

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Abstract

In this paper two things are done. We first prove that an arbitrary power p of the Schrödinger Lagrangian in arbitrary dimension always enjoys the non-relativistic conformal symmetry. The implementation of this symmetry on the dynamical field involves a phase term as well as a conformal factor that depends on the dimension and on the exponent. This non-relativistic conformal symmetry is shown to have its origin on the conformal isometry of the power p of the Klein–Gordon Lagrangian in one higher dimension which is related to the phase of the complex scalar field.

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1. Introduction

When a law of physics does not change against some transformations, the system is said to exhibit some symmetries. The symmetries of a physical system are given by the transformations that do not change the mathematical structure of the system. One can even start by defining the transformations and then to find the mathematical structure compatible with these transformations. The determination of the symmetries of a system can also be a powerful instrument since it may allow us to put the problem into a simpler form or it can permit us to obtain nontrivial solutions from trivial ones. It is then clear that the problem of the identification of the symmetries underlying an equation is not an academic question but rather a fundamental one. In this paper we shall be concerned with the Schrödinger symmetry that is defined as the dynamical symmetry leaving invariant the free Schrödinger equation (see [1], [2] and [3]). The Schrödinger invariance has been relevant in a wide variety of situations as celestial mechanics [4], non-relativistic field theory [5, 6], non-relativistic quantum mechanics [7], hydrodynamics [8–11], in the context of the AdS/CFT correspondence [12] as well as in statistical physics [13].

The Schrödinger group is defined as the largest group of spacetime transformations which leave invariant the free Schrödinger equation

$$\left[i\partial_t + \frac{1}{2}\Delta_d \right] \Phi = 0, \quad (1)$$

where the operator Δ_d represents the Laplacian in d spatial dimensions. In $d + 1$ dimensions, the Schrödinger group is a $([d(d + 3) + 6]/2)$ -dimensional Lie group which can be viewed as the semidirect sum of the static Galilei group with the $SL(2, \mathbb{R})$ group. The static Galilei group which is a $(d(d + 3)/2)$ -parameter group induces the static Galilei transformations given by

$$t \rightarrow t, \quad \vec{x} \rightarrow \mathcal{R}\vec{x} + \vec{\chi} - \vec{v}t, \quad (2)$$

where $\mathcal{R} \in SO(d)$, $\vec{\chi} \in \mathbb{R}^d$ and $\vec{v} \in \mathbb{R}^d$ generate respectively the rotations, the spatial translations and the Galilean boosts. On the other hand, $SL(2, \mathbb{R})$ is the group which induces the following transformations,

$$t \rightarrow \tilde{t} = \frac{\alpha t + \beta}{\gamma t + \delta}, \quad \vec{x} \rightarrow \vec{\tilde{x}} = \frac{\vec{x}}{\gamma t + \delta}, \quad (3)$$

and is a three-parameter group since the parameters are tied by the relation $\alpha\delta - \beta\gamma = 1$. These transformations include the time translations ($\gamma = 0, \alpha = \delta = 1$), the dilatations ($\beta = \gamma = 0$) and the special conformal transformations also called the expansions ($\alpha = \delta = 1, \beta = 0$). The static Galilei transformations and time translations induce the following field change,

$$\tilde{\Phi}(t, \vec{x}) = \exp\left(i(\vec{x} \cdot \vec{v} - \frac{|\vec{v}|^2 t}{2})\right) \Phi(t + \beta, \mathcal{R}\vec{x} + \vec{\chi} - \vec{v}t), \quad (4)$$

while the scalar field changes under the dilatation as

$$\tilde{\Phi}(t, \vec{x}) = \alpha^{d/2} \Phi(\alpha^2 t, \alpha \vec{x}), \quad (5)$$

and under the special conformal transformations as

$$\tilde{\Phi}(t, \vec{x}) = \frac{1}{(1 + \gamma t)^{d/2}} \exp\left(\frac{i\gamma |\vec{x}|^2}{2(1 + \gamma t)}\right) \Phi\left(\frac{t}{1 + \gamma t}, \frac{\vec{x}}{1 + \gamma t}\right). \quad (6)$$

We remark that, in the case of the dilatation and the expansion, the transformation of the dynamical field is associated with a multiplicative factor given by $J^{\frac{d}{2(d+2)}}$, where J is the Jacobian of the transformation linking $(\tilde{t}, \vec{\tilde{x}}) \rightarrow (t, \vec{x})$. In order to be complete, we also recall that the free Schrödinger equation (1) is derived from the following Lagrangian,

$$\mathcal{L}_S = -\frac{i}{2}(\Phi^* \partial_t \Phi - \Phi \partial_t \Phi^*) + \frac{1}{2}|\vec{\nabla} \Phi|^2, \quad (7)$$

which enjoys the Schrödinger symmetry as well.

The plan of the paper is organized as follows. In the next part, we show that an arbitrary power p of the Schrödinger Lagrangian (7) in $d + 1$ dimensions also enjoys the non-relativistic conformal symmetry. The associated Noether conserved quantities are derived and are shown to reduce to the standard Schrödinger quantities for $p = 1$. The origin of this non-relativistic conformal symmetry is explained in section 3 using a Kaluza–Klein-type framework in one higher dimension; this extra dimension is related to the phase of the complex scalar field. More precisely, we define a $(d + 2)$ -dimensional Minkowski spacetime endowed with a covariantly constant and lightlike vector field ξ . On this manifold, we consider the relativistic action given by the power p of the complex Klein–Gordon Lagrangian, and we show that the conformal isometries preserving the vector ξ are precisely the non-relativistic symmetries of the extended Schrödinger equation.

2. Non-relativistic conformal equation

We now consider an action in $(d + 1)$ dimensions defined as an arbitrary power p of the Schrödinger Lagrangian (7)

$$S_p = \int d^d \vec{x} dt \mathcal{L}_S^p = \int d^d \vec{x} dt \left(-\frac{i}{2}(\Phi^* \partial_t \Phi - \Phi \partial_t \Phi^*) + \frac{1}{2} |\vec{\nabla} \Phi|^2 \right)^p, \quad (8)$$

where p is a real parameter. The associated field equation reads

$$p \left(i \partial_t \Phi + \frac{1}{2} \Delta_d \Phi \right) \mathcal{L}_S^{p-1} + \frac{1}{2} p(p-1) (i \Phi (\partial_t \mathcal{L}_S) + \vec{\nabla} \Phi \cdot \vec{\nabla} \mathcal{L}_S) \mathcal{L}_S^{p-2} = 0, \quad (9)$$

and reduces to the standard Schrödinger equation for $p = 1$. For later convenience, we derive from (9) the continuity-like equation given by

$$\partial_t (|\Phi|^2 \mathcal{L}_S^{p-1}) + \vec{\nabla} \cdot \left(\frac{-i}{2m} (\Phi^* \vec{\nabla} \Phi - \Phi \vec{\nabla} \Phi^*) \mathcal{L}_S^{p-1} \right) = 0. \quad (10)$$

We now show that, for any arbitrary value of $p \neq 0$, the extended equation (9) or equivalent action (8) possesses the Schrödinger symmetry. Firstly, the spacetime transformations leaving invariant the extended equation are given by the usual Schrödinger transformations (2) and (3). The main difference lies in the implementation of the conformal transformations on the dynamical field Φ . Indeed, the action of the static Galilei transformations and time translations is the same as in the standard Schrödinger case (4), but the scalar field changes under the dilatation ($\gamma = 0$) and the special conformal transformation ($\alpha = 1$) as

$$\tilde{\Phi}(t, \vec{x}) = \left(\frac{\alpha}{1 + \gamma t} \right)^{\frac{d+2-2p}{2p}} \exp \left(i \frac{\gamma |\vec{x}|^2}{2(1 + \gamma t)} \right) \Phi \left(\frac{\alpha^2 t}{\gamma t + 1}, \frac{\alpha \vec{x}}{\gamma t + 1} \right). \quad (11)$$

This means that for a scalar field $\Phi(t, \vec{x})$ solving the field equation (9), the transformed field $\tilde{\Phi}(t, \vec{x})$ defined by (11) also verifies the equation for any value of the parameter p . The same conclusion can be obtained by observing that under a dilatation or a special conformal transformation (11), the Schrödinger Lagrangian rescales as

$$\mathcal{L}_S \rightarrow \left(\frac{\alpha}{1 + \gamma t} \right)^{(d+2)/p} \mathcal{L}_S,$$

and hence the power p of the Schrödinger Lagrangian exactly compensates the Jacobian of the conformal transformations which means that action (8) remains unchanged $S_p \rightarrow S_p$. A direct application of the Noether theorem yields the following constants of motion,

$$\begin{aligned} H &= \int d^d \vec{x} \mathcal{H} = \int d^d \vec{x} \left[\frac{p}{2} |\vec{\nabla} \Phi|^2 \mathcal{L}_S^{p-1} \right], \\ \vec{P} &= \int d^d \vec{x} \vec{P} = \int d^d \vec{x} \left[-\frac{ip}{2} (\Phi^* \vec{\nabla} \Phi - \Phi \vec{\nabla} \Phi^*) \mathcal{L}_S^{p-1} \right], \\ M_{ij} &= \int d^d \vec{x} (x_i \mathcal{P}_j - x_j \mathcal{P}_i), \\ \vec{G} &= t \vec{P} - p \int d^d \vec{x} |\Phi|^2 \mathcal{L}_S^{p-1}, \\ D &= tH - \frac{1}{2} \int d^d \vec{x} (\vec{x} \cdot \vec{P}), \\ K &= -t^2 H + 2tD + \frac{p}{2} \int d^d \vec{x} (|\vec{x}|^2 |\Phi|^2 \mathcal{L}_S^{p-1}), \end{aligned} \quad (12)$$

which correspond respectively to the energy (time translation), the momentum (space translations), the rotations, the Galilean boosts, the dilatation and the special conformal transformation. Note that the conservation of the functional K can be viewed as a consequence of the conservations of the energy and dilatation functionals together with the continuity equation (10). We also mention that more general nonlinear terms yielding Schrödinger invariant equations can also be considered; see [14] and [15].

Various comments can be made at this stage of the analysis. Firstly, for $p = 1$, all the expressions derived reduce to those associated with the standard Schrödinger theory. Secondly, the invariance of the extended action is achieved for any value of the power p . This is due to the fact that the Jacobian of the conformal transformation can always be compensated by rescaling in an appropriate way the dynamical field (11). For the particular exponent $p = (d + 2)/2$, the multiplicative factor different from the phase term in the field transformation (11) is no longer present. This means that the Jacobian of the conformal transformation is exactly compensated by the power $p = (d + 2)/2$ of the free Schrödinger Lagrangian without the necessity of rescaling the dynamical field but just by operating a phase change. In the next section, we will explain the origin of this extended non-relativistic Schrödinger symmetry within a higher-dimensional relativistic context.

3. Relativistic origin

The purpose of this section is to provide an explanation of the Schrödinger symmetry of the extended action using a relativistic framework in one higher dimension. The clue to this relativistic framework lies in the fact that the non-relativistic spacetime Q in $(d + 1)$ dimensions can be viewed as the quotient of a $(d + 2)$ -dimensional Lorentz manifold M by the integral curves of a covariantly constant, lightlike vector field ξ . This correspondence has been used in order to derive the Schrödinger symmetry of the standard Schrödinger equation from the relativistic conformal symmetry of the conformal wave equation (see [16], [4] and [17]).

On the manifold M we adopt the coordinate system (t, \vec{x}_d, s) where (t, \vec{x}_d) are the coordinates on Q and s is the additional coordinate, and we consider the $(d + 2)$ -dimensional action given by the power p of the Klein–Gordon Lagrangian

$$\mathcal{S}_p = \int_M \sqrt{-g} \, d^{d+2}x \left[\frac{1}{2} g^{\mu\nu} \partial_\mu \psi \partial_\nu \psi^* \right]^p. \quad (13)$$

The field equation obtained by varying this action with respect to the complex scalar field yields

$$\frac{1}{\sqrt{-g}} \partial_\mu [\sqrt{-g} \partial^\mu \psi (\partial_\sigma \psi \partial^\sigma \psi^*)^{p-1}] = 0. \quad (14)$$

On the Minkowski spacetime, we consider the flat metric written in lightcone coordinates as

$$ds^2 = d\vec{x}_d^2 + 2 \, dt \, ds \quad (15)$$

for which the covariantly constant, lightlike vector field ξ is chosen to be $\xi^\mu \partial_\mu = \partial_s$. In order to establish the correspondence with the extended Schrödinger equation (9), the field ψ is assumed to satisfy an equivariance condition given by

$$\xi^\mu \partial_\mu \psi = i\psi, \quad (16)$$

which in turn implies that the function

$$\Phi = e^{-is} \psi \quad (17)$$

is a function defined on Q since $\partial_s \Phi = 0$. It is then simple to see that in Minkowski space with metric (15), the extended wave equation (14) together with the equivariance condition (16) is

equivalent to the extended Schrödinger equation (9). The same conclusion can be achieved at the level of the actions in the sense that action (13) evaluated on the flat metric (15) for a scalar field satisfying the equivariance condition (16) reduces to the extended Schrödinger action (8). We now study the symmetries of the coupled system given by equation (14) with condition (16). We recall that a conformal isometry on the Lorentz manifold M is a diffeomorphism $\varphi : M \rightarrow M$, for which there is a nonvanishing function Ω such that $(\varphi^*g)_{\mu\nu} = \Omega^2 g_{\mu\nu}$ (see, e.g., [18]). It is simple to show that the conformal isometries that preserve the vertical vector, i.e. $\varphi^*\xi = \xi$, are symmetries of the coupled system (14) and (16) for which the implementation on the dynamical field is given by

$$\tilde{\psi} = \Omega^{\frac{2p-d-2}{2p}} \varphi^* \psi. \tag{18}$$

In the case of the flat metric (15), the ξ -preserving conformal isometries transformations form a subgroup of the conformal group and the conformal Killing vector field is given in the basis $(\partial_t, \partial, \partial_s)$ by

$$(X^\mu) = \begin{pmatrix} \chi t^2 + \delta t + \epsilon \\ \mathcal{R}\vec{x} + (\frac{1}{2}\delta + \chi t)\vec{x} + t\vec{\beta} + \vec{\gamma} \\ -\frac{1}{2}\chi|\vec{x}|^2 - \vec{\beta} \cdot \vec{x} + \eta \end{pmatrix}, \tag{19}$$

where $\mathcal{R} \in SO(d)$, $\vec{\beta}, \vec{\gamma}, \epsilon, \chi, \delta$ and η are interpreted as rotation, boost, space translation, time translation, expansion, dilatation and vertical translation. The integration of the Lie differential equation for transformation group yields the following spacetime transformations,

$$\tilde{x} = \frac{\mathcal{R}\vec{x} - \vec{v}t + \vec{\chi}}{\gamma t + \delta}, \tag{20a}$$

$$\tilde{t} = \frac{\alpha t + \beta}{\gamma t + \delta}, \tag{20b}$$

$$\tilde{s} = s + \frac{\gamma}{2} \frac{|\mathcal{R}\vec{x} - \vec{v}t + \vec{\chi}|^2}{\gamma t + \delta} + \mathcal{R}\vec{x} \cdot \vec{v} - \frac{t}{2} |\vec{v}|^2 + h, \tag{20c}$$

with the restriction $(\alpha\delta - \beta\gamma) = 1$. The corresponding conformal isometry factor is given by

$$\Omega = \Omega(t) = \gamma t + \delta. \tag{21}$$

The first two equations (20a)–(20c) correspond to the Schrödinger transformations, while the third transformation (20) possesses information concerning the phase change of the complex scalar field. Indeed, combining the equivariance condition (16)–(17) together with the law transformation of ψ (18), we have

$$\tilde{\psi}(t, \vec{x}, s) = e^{is} \tilde{\Phi}(t, \vec{x}) \implies \tilde{\Phi}(t, x) = \Omega(t)^{\frac{2p-d-2}{2p}} e^{i(\tilde{s}-s)} \Phi(\tilde{t}, \tilde{x}), \tag{22}$$

and we obtain the change field for the extended Schrödinger field (11). More precisely, from this expression, it is clear that the phase change of the Schrödinger field is associated with the change of the additional coordinate s , while the multiplicative factor is given by the conformal isometry factor (21).

To conclude, we mention that the same analysis can be done in curved spacetime by considering the following action,

$$L_p = \int \sqrt{-g} d^{d+2}x \left[\frac{1}{4} (2p - d - 2) (\psi \square \psi^* + \psi^* \square \psi) + \frac{(2p - d - 2)^2}{8p(d + 1)} R |\psi|^2 - \frac{1}{2} (d + 2) (p - 1) g^{\mu\nu} \partial_\mu \psi \partial_\nu \psi^* \right]^p, \tag{23}$$

which enjoys, for any value of the parameter p , the conformal invariance with weight given by

$$g_{\mu\nu} \rightarrow \Omega^2 g_{\mu\nu}, \quad \psi \rightarrow \Omega^{\frac{2p-d-2}{2p}} \psi. \quad (24)$$

This action generalizes the standard conformal wave action since, for $p = 1$, the conformal extended action (23) reduces after an integration by parts to the conformal wave action. It is also interesting to note that for the particular exponent $p = (d + 2)/2$ the conformal action reduce to the standard kinetic term to this power. In this case, the conformal symmetry can be viewed as the higher-dimensional extension of the two-dimensional conformal Klein–Gordon action [19]. Note that the two-dimensional situation is very special since, in this case, the conformal algebra is the direct sum of two isomorphic infinite-dimensional algebras [20].

Finally, it is also legitimate to wonder about the non-relativistic limit which is a quite difficult question [21]. In the standard case $p = 1$, it has been shown by direct computation that the resulting Lie algebra is not the Schrödinger algebra but a different algebra of same dimension and not isomorphic to the Schrödinger algebra [22]. For $p \neq 1$, the same conclusion will still be valid since the algebras involved are the same.

4. Summary and discussion

Here, we have shown that the non-relativistic conformal symmetry of the Schrödinger Lagrangian is still valid for any power of the Schrödinger Lagrangian. More precisely, the spacetime transformations leaving invariant the extended action are the usual Schrödinger transformations but the main difference lies in the implementation of the conformal transformations on the dynamical field. Indeed, this implementation is realized through a conformal factor that depends on the dimension and on the exponent as well as with a phase term. There exists a particular value of the exponent for which this conformal factor is not longer present. For a generic value of the exponent, the associated Noether conserved quantities have been obtained. The origin of this non-relativistic conformal symmetry has been analysed within a relativistic framework in one higher dimension. The same conclusions may also be valid by considering an arbitrary power of the Newton–Hooke Lagrangian because of the various analogies between both models. The main difference lies in the fact that the conformal symmetry of the free Schrödinger equation is associated with the conformal isometries that preserve the vertical vector in flat space, while in the Newton–Hooke context the metric is an homogenous plane wave metric. This is due to the fact that the Newton–Hooke group can be obtained from the (A)dS groups as the non-relativistic limit with the velocity of light c going to infinity and the cosmological constant Λ going to zero while keeping $c^2\Lambda$ finite (see, e.g., [23] and for recent work [24]).

The conformal invariance of the relativistic action (23) independently of the power p may be interesting in the search of black hole solutions. Indeed, in the standard case $p = 1$ and in four dimensions, the Einstein equations with this conformal source admits black hole solutions [25, 26]. In this example, the conformal character of the matter source has been crucial since the solution has been derived using the machinery of conformal transformations applied to minimally coupled scalar fields [26]. It would be interesting to explore whether there exist black hole solutions for the Einstein equations with the conformal source (23).

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